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A NONLINEARLY THERMOELASTIC HALF-SPACE UNDER TIME-DEPENDENT NORMAL AND SHEAR LOADING

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Abstract—A nonlinearly thermoelastic half-space is subjected to combined time-dependent normal and shear loading. The solution is obtained by a numerical method which is shown to yield accurate results by comparison with some known analytical solutions which can be obtained in some special cases. When shocks are involved, it is shown that the numerical results satisfy all the Rankine–Hugoniot jump conditions as well as the entropy condition across the shock.

INTRODUCTION

Wave propagation problems in thermoelastic media are known to be mathematically difficult due to the coupling of mechanical and thermal effects. Even in the linear theory of thermoelasticity substantial difficulties are encountered in obtaining closed form solution to the complete coupled system of equations, and usually only solutions valid for small times are obtained, see for example the review of Nowacki[1].

When finite wave propagation in heat conducting materials is considered, the equations become even much more complicated due to the presence of nonlinearity. To our knowledge the only existing solutions of the complete equations including thermal terms in nonlinear dynamic elasticity are the dilatational constant wave profile given by Bland[2] and the quasi-transverse constant profile discussed by Craine[3]. Johnson[4] applied perturbation and asymptotic methods to the problem of a nomally loaded half-space.

In [5] the authors solved the problem of a nonlinearly thermoelastic half-space subjected to time-dependent normal loading by employing a numerical method of solution. It was shown that the developed numerical scheme was accurate and reliable by comparison with analytical solutions which can be obtained in some special cases. Furthermore, the numerical method was able to handle successfully shocks which are known to occur in nonlinear problems.

In this paper the more complex and interesting problem of finite amplitude one-dimensional wave propagation in a thermoelastic half-space subjected to combined normal and shear loading is considered. It is known that in the presence of nonlinearity, normal and shear disturbances are coupled and a propagating shear disturbance cannot exist by itself in the absence of a normal disturbance. This nonlinear effect leads to a much wider range of phenomena in the propagation of waves. Such combined effects are the quasi-transverse waves, shocks and the constant wave profile. Here we generalize our previous numerical method in order to deal with the present case of three displacement components.

The accuracy and reliability of the generalized scheme is checked by comparison with the following special cases for which analytical results can be derived. (a) The quasi-transverse simple wave solution which exists in a nonconducting material[2]. (b) The quasi-transverse thermoelastic shock[3]. In this case the success of the numerical process in handling shocks is illustrated by showing that the numerical results satisfy all the Rankine–Hugoniot jump conditions as well as the entropy condition. (c) Circularly polarized thermoelastic shocks[2] for which it is shown here that the entropy condition is unconditionally satisfied. (d) Quasi-transverse constant wave profile[3]. This very interesting case in which the waves propagate steadily in a thermoelastic material without change in shape is generated here numerically.

FORMULATION OF THE PROBLEM

Consider a thermoelastic material whose specific internal energy is given in a non-dimensional form by

$$U = T_1 S + \frac{\lambda}{2\rho} I_1^2 + \frac{\mu}{\rho} I_2 - K I_1 S + \frac{\eta}{2} S^2 + \frac{\zeta}{\rho} I_2^2 \tag{1}$$

where T_1 and ρ are the non-dimensional base temperature and density respectively in the undeformed state, S is the non-dimensional specific entropy and λ, μ, K, η, ζ are material constants. In (1) I_1 and I_2 are the first and second invariants of the Green strain tensor γ_{ij}

$$\left.\begin{aligned} I_1 &= \gamma_{ii} \\ I_2 &= \gamma_{ij}\gamma_{ij} \end{aligned}\right\} \tag{2}$$

where

$$\gamma_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j} + u_{k,i}u_{k,j}), \tag{3}$$

u_i are the components of the displacement vector $\mathbf{u}$, and $u_{j,i} = (\partial u_j/\partial a_i)$ and (a_1, a_2, a_3) are the non-dimensional cartesian coordinate of the particle in the undeformed state.

We choose as in [5] the reference dimensional units $\hat{c}_0^2 = (\hat{\lambda} + 2\hat{\mu})/\hat{\rho}$, $\hat{T}_1$, $\hat{\rho}$ and $\hat{d}_0$ for the velocity, temperature, density and length respectively. The resulting dimensional quantities for the internal energy, entropy, time and material constants are given in [5].

For an internal energy given by (1) with $\zeta = 0$ we obtain the "quadratic material" for which the internal energy contains as far as second order terms in γ_{ij} and the entropy. On the other hand the quadratic material does *not* admit either quasi-transverse shock waves or the quasi-transverse constant wave profile which will be discussed in the sequel. The term with ζ in (1) is the necessary lowest order term which has to be added to the quadratic material in order that the two mentioned important phenomena exist.

We treat the one-dimensional motion of a homogeneous isotropic thermoelastic half-space $a_1 \geq 0$ so that all quantities depend on the initial position a_1 and time t. Accordingly, we obtain the following expressions for I_1 and I_2 in (2)

$$\left.\begin{aligned} I_1 &= m_1 + \tfrac{1}{2}(m_1^2 + m_2^2 + m_3^2) \equiv \Lambda(a_1, t) \\ I_2 &= \Lambda^2 + \tfrac{1}{2}(m_2^2 + m_3^2) \end{aligned}\right\} \tag{4}$$

where $m_i(a_1, t)$ are the displacement gradients $m_i = (\partial u_i/\partial a_1)$. Hence (1) reduces to:

$$\begin{aligned} U(m_i, S) = T_1 S + \frac{\lambda}{2\rho}\Lambda^2 + \frac{\mu}{\rho}\left[\Lambda^2 + \frac{1}{2}(m_2^2 + m_3^2)\right] \\ + \frac{\eta}{2}S^2 - KS\Lambda + \frac{\zeta}{\rho}\left[\Lambda^2 + \frac{1}{2}(m_2^2 + m_3^2)\right]^2. \end{aligned} \tag{5}$$

The equations of motion and heat conduction in the absence of body forces and thermal sources in the Lagrangian description are given by

$$\rho\frac{\partial^2}{\partial t^2}u_i = \frac{\partial}{\partial a_1}L_{i1} \quad (i = 1, 3) \tag{6}$$

$$\rho T\frac{\partial S}{\partial t} = -\frac{\partial Q}{\partial a_1} \tag{7}$$

where L_{i1} are the Piola–Kirchhoff stress components given by

$$L_{11} = \rho\frac{\partial U}{\partial m_1} = \rho(1+m_1)\left[c\Lambda + 2\frac{\zeta}{\rho}\Lambda(N+2\Lambda^2) - \frac{K}{\eta}\theta\right]$$

$$L_{21} = \rho\frac{\partial U}{\partial m_2} = m_2\left[\mu + \rho c\Lambda + \zeta(1+2\Lambda)N + 2\zeta\Lambda^2(1+2\Lambda) - \frac{\rho K}{\eta}\theta\right]$$

$$L_{31} = \rho\frac{\partial U}{\partial m_3} = m_3\left[\mu + \rho c\Lambda + \zeta(1+2\Lambda)N + 2\zeta\Lambda^2(1+2\Lambda) - \frac{\rho K}{\eta}\theta\right] \tag{8}$$

where

$$\left.\begin{aligned} c &= (\lambda + 2\mu)/\rho - K^2/\eta \\ N &= m_2^2 + m_3^2 \\ \theta &= T - T_1 \end{aligned}\right\} \tag{9}$$

and T is the temperature given by

$$T = \frac{\partial U}{\partial S} = T_1 + \eta S - K\left[m_1 + \frac{1}{2}(m_1^2 + N)\right]. \tag{10}$$

In (7) $Q(a_1, t)$ is the Lagrangian heat flux which, by adopting Fourier law is given by

$$Q = -k\frac{\partial T}{\partial a_1} \tag{11}$$

where k is the coefficient of heat conduction which is assumed to be constant. The dimensional expressions for the heat flux and heat conductivity are given in [5] in terms of the above chosen reference units. By examing the stress components in (8) it is obvious that up to first order terms, the classical stress–strain–temperature relations are obtained.

Using (8) in (6) we obtain after some manipulations the following equation of motion

$$\frac{\partial^2\mathbf{u}}{\partial t^2} = \mathbf{A}\frac{\partial^2\mathbf{u}}{\partial a_1^2} - \mathbf{B}\frac{\partial\boldsymbol{\theta}}{\partial t} \tag{12}$$

where $\mathbf{A}$ is the following symmetric matrix

$$A_{11} = c\Lambda + \frac{2\zeta}{\rho}(2\Lambda^3 + \Lambda N) + (1+m_1)^2\left[c + \frac{2\zeta}{\rho}(6\Lambda^2 + N)\right] - \frac{K}{\eta}\theta$$

$$A_{12} = (1+m_1)m_2R_1$$

$$A_{13} = (1+m_1)m_3R_1$$

$$A_{22} = R_2m_2^{\,2} + R_3$$

$$A_{23} = m_2m_3R_2$$

$$A_{33} = A_{22}$$

with

$$R_1 = c + 2\zeta(2\Lambda + 6\Lambda^2 + N)/\rho$$

$$R_2 = c + 2\zeta(4\Lambda + 6\Lambda^2 + N + 1)/\rho$$

$$R_3 = c\Lambda + \mu/\rho + \zeta N/\rho + 2\zeta(\Lambda N + \Lambda^2 + 2\Lambda^3)/\rho - \frac{K}{\eta}\theta.$$

The matrix **B** is given by

$$\mathbf{B} = \frac{K}{\eta}\begin{bmatrix} 1+m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{13}$$

and $\boldsymbol{\theta}$ is the vector whose components are θ.

The heat conduction equation (7) and (10) and (11) takes the form

$$(T_1+\theta)\frac{\partial\theta}{\partial t} = \frac{k\eta}{\rho}\frac{\partial^2\theta}{\partial a_1^{\,2}} - K(T_1+\theta)\left[(1+m_1)\frac{\partial^2 u_1}{\partial a_1\partial t} + m_2\frac{\partial^2 u_2}{\partial a_1\partial t} + m_3\frac{\partial^2 u_3}{\partial a_1\partial t}\right]. \tag{14}$$

From thermodynamic considerations it can be shown[2] that

$$\lambda + 2\mu/3 > 0, \quad \mu > 0, \quad \lambda - \rho K^2/\eta + 2\mu/3 > 0. \tag{15}$$

At $t = 0$, the half-space $a_1 \geq 0$ is described by the initial conditions

$$\left.\begin{aligned} \mathbf{u}(a_1, t) &= \mathbf{u}_0(a_1) \\ \frac{\partial \mathbf{u}(a_1, t)}{\partial t} &= \mathbf{v}_0(a_1) \\ \theta(a_1, t) &= \theta_0(a_1) \end{aligned}\right\} \text{ at } t = 0, \tag{16}$$

and for $t > 0$ it is subjected at $a_1 = 0$ to the boundary conditions

$$\left.\begin{aligned} m_i(0, t) &= f_i(t), \quad i = 1, 3, \\ \theta(0, t) &= g(t) \end{aligned}\right\} \tag{17}$$

where $f_l(t)$, $g(t)$ are prescribed time functions. The hyperbolic-parabolic system of nonlinear equations (12) and (14) together with the initial and boundary conditions (16–17) and the boundedness of the displacements and temperature as $\mathbf{a} \to \infty$ govern completely the subsequent motion for $t > 0$.

FINITE-DIFFERENCE FORMULATION

A finite-difference formulation is given herein in order to solve the above nonlinear system together with its initial and boundary conditions. We shall show, by comparison with some analytical conclusions which could be drawn in some special circumstances, that the obtained numerical solutions are excellent. The present formulation is a direct generalization of that given in [5] and consequently we shall present it briefly.

We introduce the space divisions Δa_1 and time intervals Δt. Let

$$h_i^n \equiv h(i\Delta a_1, n\Delta t) = h(a_1, t) \qquad i, n = 0, 1, 2, \ldots \tag{18}$$

The system of equations (12) can be solved by an explicit approximation which yields the displacements at the time step $t + \Delta t$ as follows

$$\mathbf{u}_i^{n+1,0} = 2\mathbf{u}_i^n - \mathbf{u}_i^{n-1} + (\Delta t)^2 L[\mathbf{u}_i^n, \theta_i^n] \tag{19}$$

where L is a spatial difference operator which expresses the discretized version of the differential expression on the right hand side of (12). Furthermore at each time step an iterative process is applied in the form

$$\mathbf{u}_i^{n+1,j} = 2\mathbf{u}_i^n - \mathbf{u}_i^{n-1} + (\Delta t)^2\{w_3 L[\mathbf{u}_i^{n+1,j-1}, \theta_i^n] + w_2 L[\mathbf{u}_i^n, \theta_i^n] + w_1 L[\mathbf{u}_i^{n-1}, \theta_i^{n-1}]\}/(w_1 + w_2 + w_3) \tag{20}$$

where j is the number of the iteration, $j = 1, 2, \ldots$, and w_l are weight numbers. This iterative procedure is applied in order to remove numerical oscillations which are known to occur near shock waves, see [5] for a more detailed discussion.

As to the nonlinear heat conduction equation (14), it is solved implicitly yielding the following system of algebraic equations at each time step, in the unknowns θ_i^{n+1} $(i = 1, 2, \ldots, M)$.

$$-\epsilon\theta_{i-1}^{n+1} + (2\epsilon + \theta_i^n + T_1)\theta_i^{n+1} - \epsilon\theta_{i+1}^{n+1} = \epsilon(\theta_{i+1}^n + \theta_{i-1}^n) + \theta_i^n(\theta_i^n + T_1 - 2\epsilon) + \Delta t \, . \, r_i^n \tag{21}$$

where

$$\epsilon = k\eta\Delta t/2\rho(\Delta a_1)^2 \tag{22}$$

$$r_i^n = -K(\theta_i^n + T_1)[(1 + \bar{m}_1)\bar{m}_{1i}^n + \bar{m}_{2i}^n\bar{\bar{m}}_{2i}^n + \bar{m}_{3i}^n\bar{\bar{m}}_{3i}^n] \tag{23}$$

and $\bar{m}_{li}^n$, $\bar{\bar{m}}_{li}^n$ $(l = 1, 3)$ are the central difference approximations of $(\partial u_l/\partial a_1)$, $(\partial^2 u_l/\partial a_1 \partial t)$ respectively. In these equations $M = a_1^\dagger/\Delta a_1$ with $a_1^\dagger$ being a point within the half-space far enough from the boundary such that the values of the displacements and temperature have no influence for a preassinged degree of accuracy on their values at smaller distances $a_1 < a_1^\dagger$ for a given range of space and time. For more details regarding the numerical formulation see [5].

All results in this paper are given for the mesh space increment $\Delta a_1 = 0{\cdot}01$ with $j = 1$ in (20),

$M = 400$ in (21) and the weight factors $w_1 = -1$, $w_2 = 10$, $w_3 = 0$. Apart from the quasi-transverse constant wave profile treated in the last section, the following material constants are chosen

$$\lambda = \tfrac{1}{3}, \quad \mu = \tfrac{1}{3}, \quad K = 1, \quad \eta = 2, \quad k = 1.$$

Clearly this choice satisfies the inequalities (15).

THE EFFECT OF HEAT CONDUCTION ON QUASI-TRANSVERSE SIMPLE WAVES

Let us consider the limiting situation of a nonconducting material for which $k = 0$ (adiabatic case). It is known[2] that the corresponding solution to this case can be obtained from the isentropic situation, for which $S \equiv 0$, as long as shocks are absent. On the other hand, the solution to the isentropic problem is obtained by solving the system of equations (12) after setting $K = 0$ which corresponds to a nonlinearly elastic medium. Consequently we consider an elastic half-space $a_1 \geq 0$ at rest for $t \leq 0$ with normal and shear loading on its boundary, at $t > 0$, as follows

$$\begin{aligned} m_1(0, t) &= -\frac{\lambda + 2\mu}{2(\lambda + \mu)}[\gamma f(t)]^2 \\ m_2(0, t) &= \gamma f(t) \\ m_3(0, t) &= 0 \end{aligned} \tag{24}$$

where γ is a positive coefficient and $f(t)$ is a smooth rising function of time given by

$$f(t) = [t^2 H(t) - 2(t - \tau)H(t - \tau) + (t - 2\tau)^2 H(t - 2\tau)]/2\tau^2 \tag{25}$$

with $H(t)$ being the Heaviside step function. This function rises smoothly from zero at $t = 0$ up to 1 at $t = 2\tau$ where τ is a chosen parameter.

It can be shown[2] that with (24) the nonlinear elastic equations (12) (with $K = 0$) admit, for small amplitudes (but not infinitesimal), the quasi-transverse simple wave solution given by

$$\begin{aligned} m_1(a_1, t) &= -\frac{\lambda + 2\mu}{2(\lambda + \mu)}\left[\gamma f\left(t - \frac{a_1}{v(m_2)}\right)\right]^2 \\ m_2(a_1, t) &= \gamma f\left(t - \frac{a_1}{v(m_2)}\right) \\ m_3(a_1, t) &= 0 \end{aligned} \tag{26}$$

where $v(m_2)$ is the velocity of the acceleration wave given by

$$v(m_2) = \left(\frac{\mu}{\rho}\right)^{1/2}[1 + 3\psi m_2^2/2\mu] \tag{27}$$

with

$$\psi = \zeta - \frac{\mu}{2} - \frac{\mu^2}{2(\lambda + \mu)} \tag{28}$$

and subject to the condition $\psi < 0$. This condition ensures that shocks will not form in the present case of loading conditions at the boundary.

The quasi-transverse simple wave (26) provides a direct analytical check to the numerical scheme in the special case of $k = 0$. In Fig. 1 the analytical and numerical solutions for the normal and tangential displacement gradients are shown at the station $a_1 = 0{\cdot}3$. The applied load rise time is $2\tau = 0{\cdot}2$ and $\zeta = 0$ which obviously satisfy the condition $\psi < 0$. The expressions for $m_1(a_1, t)$ in (26) and $v(m_2)$ in (27) are in an error of γ^4 which is very small for our present choice of $\gamma = 0{\cdot}25$. The obtained numerical and analytical solutions are in excellent agreement and are up to the scale of the plot indistinguishable.

For a heat conducting half-space (with $k = 1$) the corresponding numerical solution for the displacement gradients as well as the entropy, temperature and heat flux are shown at Fig. 1 at the station $a_1 = 0{\cdot}3$, with the same boundary conditions (24) and $\theta = 0$ at $a_1 = 0$. This figure exhibits very well the effect of heat conduction on the various dependent variables in the present situation of smooth loading.

QUASI-TRANSVERSE THERMOELASTIC SHOCKS

In the present section solutions which contain propagating shocks in a thermoelastic half-space are considered. For a finite conductivity, $k \neq 0$, the propagation of shocks are governed by the following jump conditions for the momentum, energy and compatibility across a shock[2]

$$\left[\frac{\partial U}{\partial m_i}\right] = V^2[m_i] \tag{29}$$

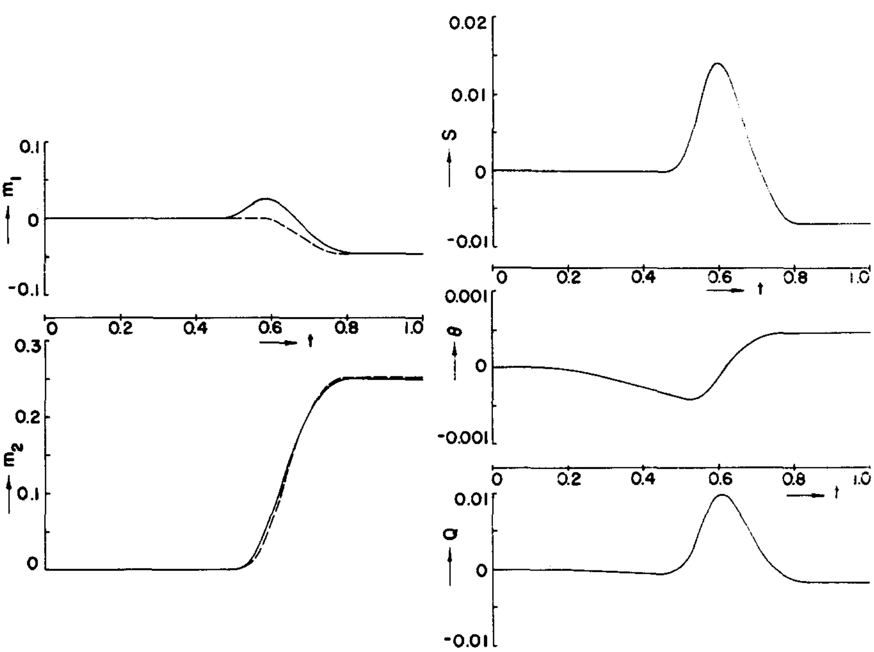

Fig. 1. Quasi-transverse simple waves (dashed line) at station $a_1 = 0{\cdot}3$ in a nonconducting half-space with the boundary conditions (24) ($\gamma = 0{\cdot}25$ and $2\tau = 0{\cdot}2$). The analytical and numerical solutions coincide. The solid lines show at the same station the numerical solution for the displacement gradients, entropy, temperature and heat flux for a conducting half-space ($k = 1$) with the boundary conditions (24) at the base temperature.

$$\frac{1}{\rho}[Q] = V[U] + \frac{1}{2}V[v_i^2] + \left[v_i \frac{\partial U}{\partial m_i}\right] \tag{30}$$

$$V[m_i] + [v_i] = 0 \tag{31}$$

where $v_i = (\partial u_i/\partial t)$ is the particle velocity, V is the propagation velocity of the shock and $[h] = h^- - h^+$, h^+ and h^- being the values of h must ahead and just behind the discontinuity, respectively. To these conditions we must add the entropy condition

$$\rho V[S] \geq \left[\frac{Q}{T}\right] \tag{32}$$

as well as the condition that shocks in a conducting solid governed by the Fourier law (11) are necessarily isothermal, i.e.

$$[T] = 0. \tag{33}$$

For shock waves propagating in a thermoelastic medium which is previously undeformed at the base temperature T_1 Craine [3] obtained the following solution for a quasi-transverse shock of small amplitudes in a_1, a_2 directions

$$m_1^* = -0{\cdot}5[(\lambda + 2\mu - \rho K^2/\eta)/(\lambda + \mu - \rho K^2/\eta)](m_2^*)^2[1 + 0(m_2^*)^2] \tag{34}$$

and

$$V^2 = \frac{\mu}{\rho} + \frac{1}{4\rho}\left[\lambda + 2\mu + 2\zeta - \frac{\rho K^2}{\eta} - \left(\lambda + 2\mu - \frac{\rho K^2}{\eta}\right)^2 \Big/ \left(\lambda + \mu - \frac{\rho K^2}{\eta}\right)\right](m_2^*)^2 . [1 + 0(m_2^*)^2] \tag{35}$$

where m_i^* denotes the final value of m_i. In addition, the entropy condition (32) yields the following necessary and sufficient condition for the existence of a quasi-transverse shock in a thermoelastic medium

$$\lambda + 2\mu + 2\zeta - \rho K^2/\eta - (\lambda + 2\mu - \rho K^2/\eta)^2/(\lambda + \mu - \rho K^2/\eta) > 0. \tag{36}$$

This inequality imposes a stringent condition on the material if it is to admit such a shock, and for $\zeta = 0$ (36) is violated. On the other hand for our choice of the material constants this condition is satisfied for $\zeta = 1$.

In order to produce the quasi-transverse shock in the thermoelastic half-space which is initially at rest at the base temperature T_1, we apply the following boundary conditions at $a_1 = 0$

$$\begin{aligned} m_2(0, t) &= \gamma H(t) \\ m_1(0, t) &= -0{\cdot}5[(\lambda + 2\mu - \rho K^2/\eta)/(\lambda + \mu - \rho K^2/\eta)]\gamma^2 H(t) \\ m_3(0, t) &= 0 \\ \theta(0, t) &= 0 \end{aligned} \tag{37}$$

with $\gamma = 0{\cdot}25$.

In Fig. 2 numerical solutions to the displacement gradients, particle velocities, entropy, temperature and heat flux are shown at station $a_1 = 0{\cdot}3$ for the two different cases $\zeta = 0$ and $\zeta = 1$. Whereas for $\zeta = 0$ smooth solutions are obtained, the case with $\zeta = 1$ yields in contrast propagating shock waves in accordance with the above analytical considerations.

Let us check now the fulfillment of the jump conditions (29–31) and the entropy condition (32) across the shock. A direct measurement of the velocity from the arrival time of the shock at station $a_1 = 0{\cdot}3$ yields the value $V = 0{\cdot}6$ whereas the predicted value according to (35) is 0·58. Furthermore, $[m_1] = -0{\cdot}091$, $[m_2] = 0{\cdot}25$, $[v_1] = 0{\cdot}054$, $[v_2] = -0{\cdot}15$ and $[S] = -0{\cdot}0275$. For these values we obtain $[\partial U/\partial m_1] = -0{\cdot}0323$ as against $V^2[m_1] = -0{\cdot}0327$, and $[\partial U/\partial m_2] = 0{\cdot}0913$ as against $V^2[m_2] = 0{\cdot}09$ showing excellent fulfillment of (29). Similarly the compatibility conditions (31) are well satisfied. As to the energy condition (30), we obtain again by a direct measurement that $[Q] = -0{\cdot}015$, whereas the right hand side of (30) yields the value $-0{\cdot}017$. For the entropy condition (32) we have $VT[S] = -0{\cdot}015$ showing that, up to the accuracy of the measuring procedure, (32) holds as equality. Figure 2 shows also clearly that there is no jump in the temperature across the shock, i.e. $[T] = 0$ in agreement with (33). We conclude, therefore, that the resulting numerical solution yields the correct jumps across the shock according to the above conditions.

CIRCULARLY POLARIZED THERMOELASTIC SHOCKS

For an isotropic thermoelastic material, the internal energy in the case of one-dimensional wave propagation has the general dependence form $U(m_1, N, S)$ where N was defined in (9).

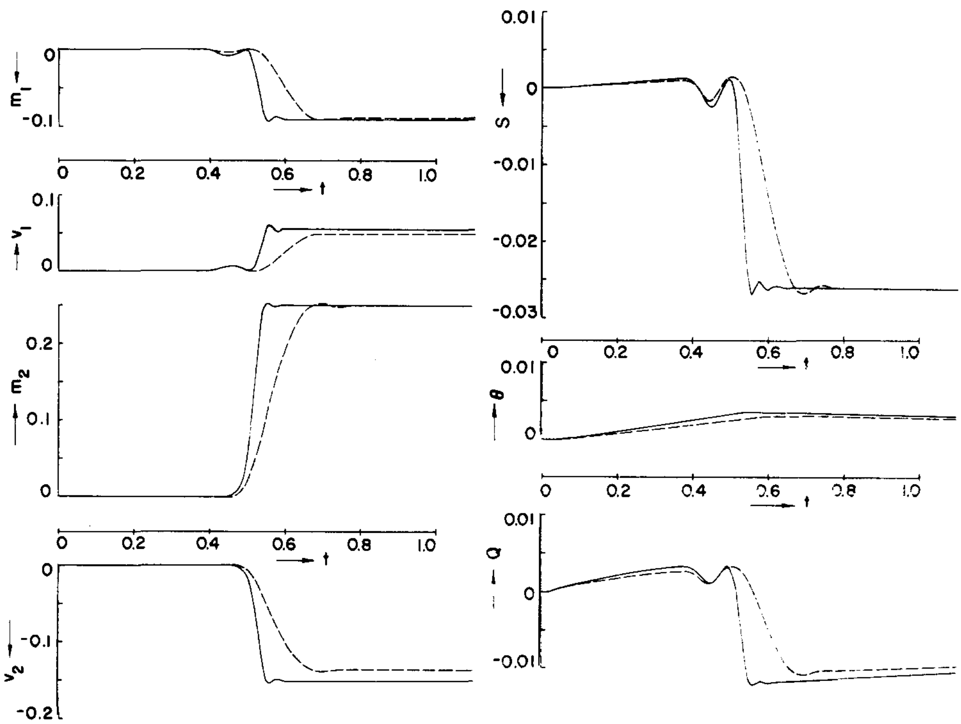

Fig. 2. Quasi-transverse wave (dashed line, $\zeta = 0$) and shock (solid line, $\zeta = 1$) at station $a_1 = 0{\cdot}3$ in a thermoelastic half-space subjected to the boundary conditions (37) with $\gamma = 0{\cdot}25$.

Suppose that m_1, m_2, m_3 and T are given ahead of the shock. Then since $[T]=0$ across a shock, equations (29) with (10) are three equations for the four unknowns m_1, m_2, m_3 behind the shock and V. By considering a situation for which

$$\left[\frac{\partial U}{\partial N}\right]=0, \tag{38}$$

Bland[2] obtained the solution

$$V^2=2\frac{\partial U}{\partial N} \tag{39}$$

and

$$\left.\begin{aligned}[m_1]=0\\ [N]=0\end{aligned}\right\}. \tag{40}$$

Hence m_1, N are continuous across the shock represented by (38) and allow only the direction of the component of m_2, m_3 to be discontinuous, thus obtaining a circularly polarized shock propagating with the velocity (39) in a thermoelastic material.

We shall show now the interesting property that for a circularly polarized shock propagating in a thermoelastic material the jump of the heat flux across the shock is zero, i.e.

$$[Q]=0. \tag{41}$$

For this purpose let us apply the following identity

$$[ab]=[a][b]+a^+[b]+b^+[a] \tag{42}$$

which yields together with the compatibility conditions (31) the following expression for $[v_i^2]$ in (30)

$$[v_i^2]=V^2[m_i]^2-2V[m_i]v_i^+. \tag{43}$$

Similarly we obtain for $[v_i(\partial U/\partial m_i)]$ in (30)

$$\left[v_i\frac{\partial U}{\partial m_i}\right]=-V^3[m_i]^2+V^2[m_i]v_i^+-V[m_i]\left(\frac{\partial U}{\partial m_i}\right)^+. \tag{44}$$

Hence, (30) takes the form

$$\frac{1}{\rho}[Q]=V[U]-\frac{1}{2V}\left[\left(\frac{\partial U}{\partial m_i}\right)^2\right]. \tag{45}$$

But for the circularly polarized shock in a heat conducting solid, it is obvious from (10), (33) and (40) that the shock is isentropic, i.e. $[S]=0$, which imply together with (40) that $[U]=0$. Furthermore, for this shock

$$\left[\left(\frac{\partial U}{\partial m_l}\right)^2\right] = \left[\left(\frac{\partial U}{\partial m_1}\right)^2\right] + \left[\left(\frac{\partial U}{\partial m_2}\right)^2\right] + \left[\left(\frac{\partial U}{\partial m_3}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial U}{\partial N}\right)^2 m_2^2\right] + 4\left[\left(\frac{\partial U}{\partial N}\right)^2 m_3^2\right]$$

$$= 4\left[\left(\frac{\partial U}{\partial N}\right)^2 N\right]$$

$$= 4\left\{\left[\left(\frac{\partial U}{\partial N}\right)^2\right] + \left(\left(\frac{\partial U}{\partial N}\right)^2\right)^+\right\}[N] + 4N^+\left[\left(\frac{\partial U}{\partial N}\right)^2\right]$$

$$= 0.$$

Therefore $[Q] = 0$.

Another interesting property of this shock in a heat conducting material is that the entropy condition (32) is always satisfied. Indeed with (33), (41) and $[S] = 0$, the inequality (32) is unconditionaly satisfied.

Let us produce now a circularly polarized shock by applying the following initial and boundary conditions for the thermoelastic half-space

$$\left.\begin{aligned} m_1(a_1, t) &= 0{\cdot}2 \\ m_2(a_1, t) &= 0{\cdot}3 \\ m_3(a_1, t) &= 0{\cdot}1 \\ \theta(a_1, t) &= 0 \end{aligned}\right\} t = 0 \tag{46}$$

and

$$\left.\begin{aligned} m_1(0, t) &= 0{\cdot}2 \\ m_2(0, t) &= 0{\cdot}1 \\ m_3(0, t) &= 0{\cdot}3 \\ \theta(0, t) &= 0 \end{aligned}\right\} t > 0. \tag{47}$$

In Fig. 3 the numerical solution for the displacement gradients and the entropy are shown at station $a_1 = 0{\cdot}3$ for $\zeta = 0$. According to (40) it is expected that $m_1(a_1, t) = 0{\cdot}2$. The maximum deviations, of the numerical solution from this value, on the other hand, is 0·01 which is obtained at $t = 0{\cdot}45$. The maximum temperature deviation from the base temperature T_1 is 10^{-4}, and the maximum value of $|Q|$ is 0·002 indicating that θ and Q are practically zero. Furthermore, the velocity of the shock according to (39) is predicted to be $V = 0{\cdot}68$, whereas by a direct measuring we obtain 0·66. We can conclude, therefore, that the circularly polarized shock is obtained with an excellent agreement with the previous analytical considerations and serves as a direct test to the accuracy and reliability of the proposed numerical scheme when shocks are involved.

QUASI-TRANSVERSE CONSTANT WAVE PROFILE

As another check to the accuracy of the numerical scheme, it will be applied in order to produce the interesting case of the quasi-transverse constant wave profile which, if it exists, it propagates steadily within a nonlinearly thermoelastic material without a change in shape or

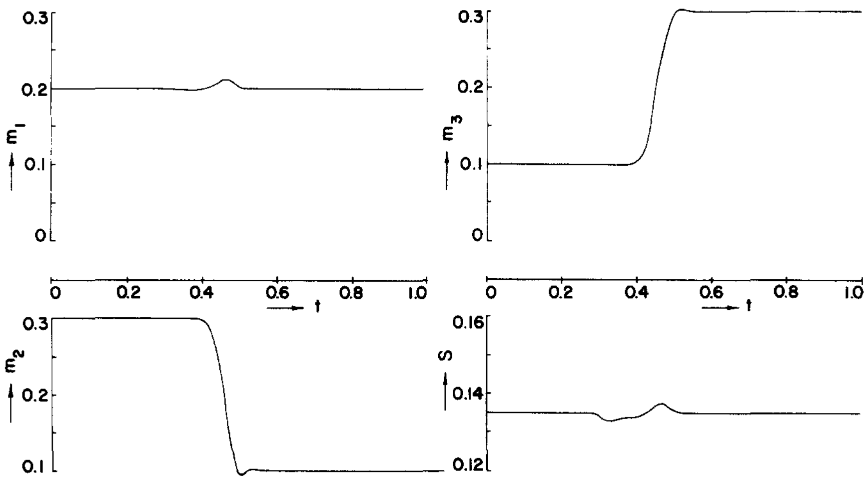

Fig. 3. Displacement gradients and entropy for a circularly polarized shock at station $a_1 = 0{\cdot}3$ within a thermoelastic half-space with the initial and boundary conditions (46) and (47) respectively.

magnitude at a constant velocity. This wave was treated by Craine [3] who showed that it consists of two separate parts, a dilatational and transverse part, which are joined together.

Let $\alpha = a_1 - Vt$ where V is the constant velocity of the wave, and without loss of generality let the axis be oriented such that $m_3 \equiv 0$.

In the dilatational part $\alpha_0 < \alpha < \infty$ and

$$\left.\begin{array}{ll} m_1 = m_1^{(D)}(\alpha), & m_2 = m_2^{(D)}(\alpha) = 0 \\ S = S^{(D)}(\alpha), & \theta = \theta^{(D)}(\alpha) \end{array}\right\} \tag{48}$$

such that

$$m_1^{(D)} = 0, \quad \theta^{(D)} = 0 \tag{49}$$

as $\alpha \to \infty$. $\alpha = \alpha_0$ is the junction point, where the transverse part starts to exist, at which $m_1^{(D)} = m_0$ and m_0 is given by [3]

$$V^2 = 2\left(\frac{\partial U}{\partial m_2^2}\right)\bigg|_{m_1 = m_0,\, m_2 = 0.} \tag{50}$$

For a smooth constant profile not containing a shock, the equation that determines the variation of the displacement gradient across the wave in this part, i.e. its variation from $m_1^{(D)} = 0$ in the initial state, where the medium is at rest, to its final value $m_1^{(D)} = m_0$ in the dilatational part is given by

$$-\rho V \alpha = \int^{m_1} k \frac{\partial \theta}{\partial m_1} \Big/ [U - 0{\cdot}5 V^2 m_1^2] \, \mathrm{d}m_1 \tag{51}$$

which yields an equation for $m_1^{(D)}$ in term of α.

In the transverse part $-\infty < \alpha < \alpha_0$, and

$$\left.\begin{array}{ll} m_1 = m_1^{(T)}(\alpha), & m_2 = m_2^{(T)}(\alpha), \\ S = S^{(T)}(\alpha), & \theta = \theta^{(T)}(\alpha) \end{array}\right\} \tag{52}$$

and such that

$$\left.\begin{array}{c} m_1^{(T)} = m_1^*, \quad m_2^{(T)} = m_2^*, \\ \theta^{(T)} = \theta^* \end{array}\right\} \tag{53}$$

as $\alpha \to -\infty$. The variation of the displacement gradient $m_1^{(T)}(\alpha)$ across the wave in this part is given by

$$-\rho V(\alpha - \alpha_0) = \int_{m_0}^{m_1} k \frac{\partial \theta}{\partial m_1} \Big/ [U - 0{\cdot}5 V^2(m_1^2 + m_2^2)]\, \mathrm{d}m_1. \tag{54}$$

For small (but not infinitesimal) displacement gradients Craine [6] obtained the following solution in the dilatational part

$$m_1^{(D)}(\alpha) = \delta m_1^* \exp[-p(\alpha - \alpha_0)] \tag{55}$$

$$S^{(D)}(\alpha) = (\lambda + \mu) m_1^{(D)}(\alpha)/\rho K \tag{56}$$

where

$$p = \rho T_1 V(\lambda + \mu)/[k\eta(\lambda + \mu - \rho K^2/\eta)] \tag{57}$$

and

$$\delta = [(\lambda + 2\mu)^2 - (\lambda + \mu)(\lambda + 2\mu + 2\zeta)]/[\mu(\lambda + 2\mu)].$$

From (55) it follows that

$$u_1^{(D)}(\alpha) = -m_1^{(D)}(\alpha)/p. \tag{58}$$

Obviously, conditions (15) yields that $p > 0$ so that (49) are satisfied.

The velocity of the constant wave profile can be shown to be

$$V^2 = \frac{1}{\rho}\left\{\mu + \left(\zeta - \frac{\mu}{2} - \frac{\mu^2}{2(\lambda + \mu)}\right)(m_2^*)^2 + 0((m_2^*)^4)\right\} \tag{59}$$

and m_1^*, m_2^* are related by

$$m_1^* = -0{\cdot}5 \frac{\lambda + 2\mu}{\lambda + \mu}(m_2^*)^2 + 0((m_2^*)^4). \tag{60}$$

In the transverse part the corresponding solution is

$$m_1^{(T)}(\alpha) = m_1^*\{1 + (\delta - 1)\exp[-q(\alpha - \alpha_0]\} \tag{61}$$

$$m_2^{(T)}(\alpha) = m_2^* y \tag{62}$$

$$S^{(T)}(\alpha) = \delta(\lambda + \mu)m_1^{(T)}(\alpha)/\rho K \tag{63}$$

where

$$y = \{1 - \exp[-q(\alpha - \alpha_0)]\}^{1/2}$$

and

$$q = \frac{\rho V T_1}{k\eta}\left\{1 - \frac{\rho K^2}{\eta}\frac{2\zeta - \mu}{(2\zeta - \mu)(\lambda + \mu) - \mu^2}\right\}^{-1}. \tag{64}$$

It follows that a necessary condition for the existence of the constant wave profile is that $q < 0$, which imposes quite strigent conditions on the parameters of the material. These conditions ensure also that V^2 in (59) is real. In order that $u_1^{(D)}(\alpha)$ in (58) passes to $u_1^{(T)}(\alpha)$ continuously at $\alpha = \alpha_0$, it follows from (61) that

$$u_1^{(T)}(\alpha) = m_1^*\left\{\alpha - \frac{\delta - 1}{q}\exp[-q(\alpha - \alpha_0)] - \frac{\delta}{p} - \alpha_0 + \frac{\delta - 1}{q}\right\}. \tag{65}$$

For $u_2^{(T)}(\alpha)$ we obtain from (62)

$$u_2^{(T)}(\alpha) = -2m_2^*\left\{y + \frac{1}{2}\log\left|\frac{y - 1}{y + 1}\right|\right\}\Big/ q. \tag{66}$$

In order to produce numerically the constant wave profile in a thermoelastic half-space we choose here the following material

$$\lambda = \tfrac{1}{2}, \quad \mu = \tfrac{1}{4}, \quad K = 1, \quad \eta = 2, \quad \zeta = \tfrac{3}{16}$$

which satisfies the inequality $q < 0$, so that the propagation of the wave in the half-space is possible.

The appropriate initial conditions at $t = 0$ are

$$u_1(a_1, 0) = \begin{cases} u_1^{(D)}(\alpha) & \text{for } \alpha > \alpha_0 \\ u_1^{(T)}(\alpha) & \text{for } \alpha < \alpha_0 \end{cases}$$

$$u_2(a_1, 0) = \begin{cases} 0 & \text{for } \alpha > \alpha_0 \\ u_2^{(T)}(\alpha) & \text{for } \alpha < \alpha_0 \end{cases}$$

and

$$\frac{\partial u_i}{\partial t} = -V\frac{\partial u_i}{\partial \alpha} \qquad (i = 1, 2).$$

In addition

$$\theta(a_1, 0) = \begin{cases} \theta^{(D)}(\alpha) & \text{for} \quad \alpha > \alpha_0 \\ \theta^{(T)}(\alpha) & \text{for} \quad \alpha < \alpha_0 \end{cases}$$

where by (10), $\theta^{(D)}$, $\theta^{(T)}$ are given in terms of $S^{(D)}$ in (56) and $S^{(T)}$ in (63), respectively.

The boundary conditions at $a_1 = 0$ are

$$m_1(0, t) = \begin{cases} m_1^{(D)}(\alpha) & \alpha > \alpha_0 \\ m_1^{(T)}(\alpha) & \alpha < \alpha_0 \end{cases}$$

$$m_2(0, t) = \begin{cases} 0 & \alpha > \alpha_0 \\ m_2^{(T)}(\alpha) & \alpha < \alpha_0 \end{cases}$$

and

$$\theta(0, t) = \begin{cases} \theta^{(D)}(\alpha) & \alpha > \alpha_0 \\ \theta^{(T)}(\alpha) & \alpha < \alpha_0. \end{cases}$$

In Fig. 4 the numerical solutions for the displacement gradients and temperature are shown at station $a_1 = 0{\cdot}3$ within the half-space, for three different values of heat conductivity coefficients, namely $k = 0{\cdot}01, 0{\cdot}1, 1$. For the junction point we choose $\alpha_0 = 0{\cdot}2$. We choose also the final value $m_2^* = 0{\cdot}1$ which yields according to (60) the value $m_1^* = -0{\cdot}0066$ and according to (59) $V = 0{\cdot}5$.

A direct comparison between the numerical and analytical results shows excellent agreement between the two solutions which especially for $k = 0{\cdot}1$ and $k = 1$ are up to the scale of the plot indistinguishable. We can conclude, therefore, that the numerical results yield, in the present interesting case too, a very accurate solution.

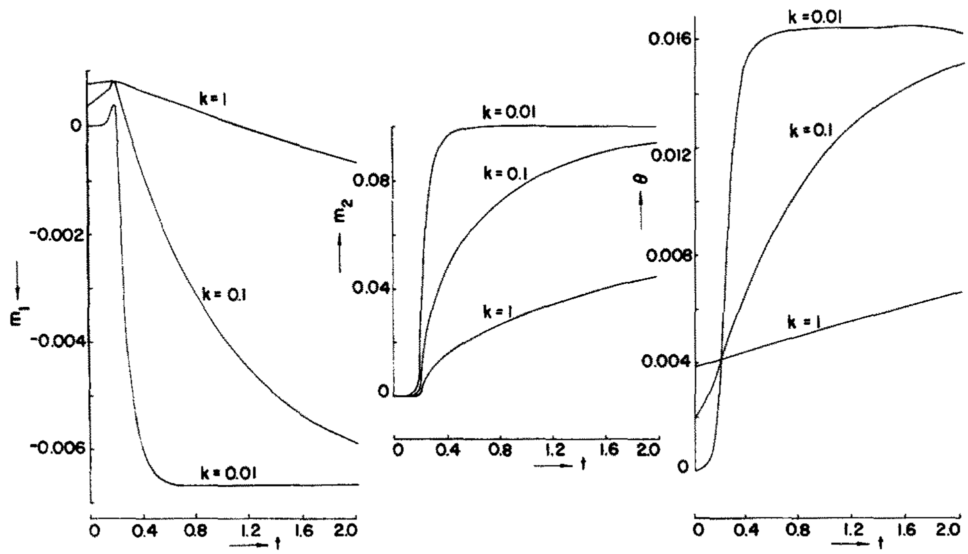

Fig. 4. Numerical solution for the displacement gradients and temperature for the constant wave profile at station $a_1 = 0{\cdot}3$ within a thermoelastic half-space, when the coefficients of heat conduction are $k = 0{\cdot}01$, $k = 0{\cdot}1$ and $k = 1$.

Fig. 4 exhibits well the effect of finite conductivity on the thickness of the propagating constant wave profile which is, as expected from the previous expressions of the various dependent variables, proportional to the heat conductivity coefficient.

CONCLUSION

A numerical scheme is presented for the solution of one-dimensional coupled nonlinear equations of motion in a thermoelastic half-space subjected to normal and shear time-dependent loading. The reliability of the method is checked and demonstrated by comparisons with some analytical conclusions which can be derived in some special cases. The present method can be applied to other unsolved thermoelastic problems. It can be also extended and applied to investigate nonlinear wave propagation in a generalized thermoelastic material which possesses a relaxation time for thermal effects appropriate for heat conduction at low temperatures, and yielding a finite thermal wave velocity. For nonlinear dilatational wave propagation this problem has been discussed recently by Beevers [7].

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